

# Disapprove of a conjecture of Erdős on primitive sequences

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## Abstract

We disapprove the conjecture of Erdős which states that for any primitive sequence  $\mathcal{A}$ , we have:

$$\sum_{a \in \mathcal{A}} \frac{1}{a \log a} \leq \sum_{p \text{ prime}} \frac{1}{p \log p}.$$

**MSC:** 11A05, 11N64

**Keywords:** Primitive sequences; Erdős conjecture; Prime numbers.

A sequence  $A$  of positive integers is called primitive if no term of the sequence divides any other. Erdős proved that for any primitive sequence  $\mathcal{A}$ , the series  $\sum_{a \in \mathcal{A}} \frac{1}{a \log a}$  converges and its sum is bounded by 1.84 (see [2]). This bound is later improved by Clark [1] to  $e^\delta$  ( $< 1.7811$ ), where  $\delta$  is the Euler constant.

In [2], Erdős conjectured that for any primitive sequence  $\mathcal{A}$ , we have:

$$\sum_{a \in \mathcal{A}} \frac{1}{a \log a} \leq \sum_{p \text{ prime}} \frac{1}{p \log p} \quad (1)$$

This means that the quantity  $\sum_{a \in \mathcal{A}} \frac{1}{a \log a}$  ( $\mathcal{A}$  is a primitive sequence) reaches its maximum value at the primitive sequence  $\mathcal{P}$  of primes numbers.

Although several results support the conjecture (see for example [3]), we prove in this paper that it is false. Actually we give a primitive sequence  $\mathcal{A}$  for which (1) doesn't hold.

Let  $\mathcal{P}$  denote the sequence of prime numbers. Set

$$\begin{aligned} \mathcal{A}_1 &:= \{pq \mid p, q \in \mathcal{P} ; p < 14 \times 10^5 \text{ and } q < 14 \times 10^5\} \\ \mathcal{A}_2 &:= \{r \in \mathcal{P} \mid r > 14 \times 10^5\} \\ \mathcal{A} &:= \mathcal{A}_1 \cup \mathcal{A}_2. \end{aligned}$$

It is clear that  $\mathcal{A}$  is a primitive sequence. The computer calculations give:

$$\sum_{a \in \mathcal{A}_1} \frac{1}{a \log a} = 1.5748 \dots$$

and

$$\sum_{p \in \mathcal{P}, p \leq 14 \times 10^5} \frac{1}{p \log p} = 1.5659 \dots$$

So, we have:

$$\sum_{a \in \mathcal{A}_1} \frac{1}{a \log a} > \sum_{p \in \mathcal{P}, p \leq 14 \times 10^5} \frac{1}{p \log p}.$$

By adding  $\sum_{a \in \mathcal{A}_2} \frac{1}{a \log a}$  to the two hand-sides of this inequality, we get:

$$\sum_{a \in \mathcal{A}} \frac{1}{a \log a} > \sum_{p \in \mathcal{P}} \frac{1}{p \log p},$$

as required.

The conjecture of Erdős is disproved. ■

## References

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